

Repeated columns and an old chestnut

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May 6, 2013

Abstract

Let $t \geq 1$ be a given integer. Let \mathcal{F} be a family of subsets of $[m] = \{1, 2, \dots, m\}$. Assume that for every pair of disjoint sets $S, T \subset [m]$ with $|S| = |T| = k$, there do not exist $2t$ sets in \mathcal{F} where t subsets of \mathcal{F} contain S and are disjoint from T and t subsets of \mathcal{F} contain T and are disjoint from S . We show that $|\mathcal{F}|$ is $O(m^k)$.

Our main new ingredient is allowing, during the inductive proof, multisets of subsets of $[m]$ where the multiplicity of a given set is bounded by $t - 1$. We use a strong stability result of Anstee and Keevash. This is further evidence for a conjecture of Anstee and Sali. These problems can be stated in the language of matrices. Let $t \cdot M$ denote t copies of the matrix M concatenated together. We have established the conjecture for those configurations $t \cdot F$ for any $k \times 2$ $(0,1)$ -matrix F .

Keywords: extremal set theory, extremal hypergraphs, $(0,1)$ -matrices, multi-set, forbidden configurations, trace, subhypergraph.

*Research supported in part by NSERC, work done while visiting the second author at USC.

†This author was supported in part by NSF grant DMS 1000475.

1 Introduction

We will be considering a problem in extremal hypergraphs that can be phrased as how many edges a hypergraph on m vertices can have when there is a forbidden subhypergraph. There are a variety of ways to define this problem (we could, but do not, restrict to (simple) k -uniform hypergraphs). We can encode a hypergraph on m vertices as an m -rowed (0,1)-matrix where the i th column is the incidence vector of the i th hyperedge. A hypergraph is *simple* if there are no repeated edges. We define a matrix to be *simple* if it is a (0,1)-matrix with no repeated columns. We will use the language of matrices in this paper.

Let M be an m -rowed (0,1)-matrix. Some notation about repeated columns is needed. For an $m \times 1$ (0,1)-column α , we define $\mu(\alpha, M)$ as the multiplicity of column α in a matrix M . We consider matrices of bounded column multiplicity. We define a matrix A to be t -*simple* if it is a (0,1)-matrix and every column α of A has $\mu(\alpha, A) \leq t$. Simple matrices are 1-*simple*. For a given matrix M , let $\text{supp}(M)$ denote the maximal simple m -rowed submatrix of M , so that if $\mu(\alpha, M) \geq 1$ then $\mu(\alpha, \text{supp}(M)) = 1$. The matrices below are a 3-simple matrix M and its support $\text{supp}(M)$.

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad \text{supp}(M) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

For two (0,1)-matrices F and A , we say that F is a *configuration* in A , and write $F \prec A$ if there is a row and column permutation of F which is a submatrix of A . Let \mathcal{F} denote a finite set of (0,1)-matrices. Let $\text{Avoid}(m, \mathcal{F}, t)$ denote all m -rowed t -simple matrices A for which $F \not\prec A$ for all $F \in \mathcal{F}$. We are most interested in cases with $|\mathcal{F}| = 1$ [5]. We do not require any $F \in \mathcal{F}$ to be simple which is quite different from usual forbidden subhypergraph problems. Our extremal function of interest is

$$\text{forb}(m, \mathcal{F}) = \max_A \{\|A\| : A \in \text{Avoid}(m, \mathcal{F}, 1)\}.$$

We find it helpful to also define

$$\text{forb}(m, \mathcal{F}, t) = \max_A \{\|A\| : A \in \text{Avoid}(m, \mathcal{F}, t)\}.$$

If $A \in \text{Avoid}(m, \mathcal{F}, t)$ then $\text{supp}(A) \in \text{Avoid}(m, \mathcal{F}, 1)$ and $\|A\| \leq t \cdot \|\text{supp}(A)\|$. We obtain

$$\text{forb}(m, \mathcal{F}) \leq \text{forb}(m, \mathcal{F}, t) \leq t \cdot \text{forb}(m, \mathcal{F}), \tag{1}$$

so that the asymptotic growth of $\text{forb}(m, \mathcal{F})$ is the same as that of $\text{forb}(m, \mathcal{F}, t)$.

We have an important conjecture about $\text{forb}(m, \mathcal{F})$. We use the notation $[M | N]$ to denote the matrix obtained from concatenating the two matrices M and N . We use the notation $k \cdot M$ to denote the matrix $[M | M] \cdots [M]$ consisting of k copies of M concatenated together. Let I_k denote the $k \times k$ *identity* matrix and let I_k^c denote the (0,1)-complement of I_k . Let T_k denote the $k \times k$ *triangular* (0,1)-matrix with the (i, j)

entry being 1 if and only if $i \leq j$. For an $m_1 \times n_1$ matrix X and an $m_2 \times n_2$ matrix Y , we define the 2-fold product $X \times Y$ as the $(m_1 + m_2) \times n_1 n_2$ matrix each column consisting of a column of X placed on a column of Y and this is done in all possible ways. This extends to p -fold products.

Definition 1.1 *Let $X(F)$ be the smallest p so that $F \prec A_1 \times A_2 \times \cdots \times A_p$ for every choice of A_i as either $I_{m/p}$, $I_{m/p}^c$ or $T_{m/p}$.*

Alternatively, assuming $F \not\prec I$ or $F \not\prec I^c$ or $F \not\prec T$, then $X(F) - 1$ is the largest choice of p so that $F \not\prec A_1 \times A_2 \times \cdots \times A_p$ for some choices of A_i as either $I_{m/p}$, $I_{m/p}^c$ or $T_{m/p}$. We note that if $A_1 \times A_2 \times \cdots \times A_p \in \text{Avoid}(m, F)$, then $\text{forb}(m, F)$ is $\Omega(m^p)$.

Details are in [5]. We are assuming m is large and divisible by p , in particular that $m \geq (k+1)(k\ell+1)$ so that $m/p \geq k\ell+1$. Divisibility by p does not affect the asymptotic growth, thus $\text{forb}(m, F)$ is $\Omega(m^{X(F)-1})$ using an appropriate $(X(F) - 1)$ -fold product.

Conjecture 1.2 [4] *Let F be given. Then $\text{forb}(m, F) = \Theta(m^{X(F)-1})$.* ■

The conjecture was known to be true for all 3-rowed F [4] and all $k \times 2$ F [3]. Section 3 shows how Theorem 1.3 establishes the conjecture for matrices $t \cdot F$ when F is a $k \times 2$ matrix. It is of interest to generalize Conjecture 1.2 to $\text{forb}(m, \mathcal{F})$ where $|\mathcal{F}| > 1$ but we know example of \mathcal{F} where the conjecture fails.

We define $F_{e,f,g,h}$ as the $(e+f+g+h) \times 2$ matrix consisting of e rows $[1\ 1]$, f rows $[1\ 0]$, g rows $[0\ 1]$ and h rows $[0\ 0]$. Let $\mathbf{1}_e \mathbf{0}_f$ denote the $(e+f) \times 1$ vector of e 1's on top of f 0's so that $F_{e,f,g,h} = [\mathbf{1}_{e+f} \mathbf{0}_{g+h} \mid \mathbf{1}_e \mathbf{0}_f \mathbf{1}_g \mathbf{0}_h]$. We let $\mathbf{1}_e$ denote the $e \times 1$ vector of e 1's and $\mathbf{0}_f$ denote the $f \times 1$ vector of f 0's. Our main result is the following which had foiled many previous attempts.

Theorem 1.3 *Let $t \geq 2$ be given. Then $\text{forb}(m, t \cdot F_{0,k,k,0})$ is $\Theta(m^k)$.*

The forbidden configuration $t \cdot F_{0,k,k,0}$ in the language of sets, consists of two disjoint k -sets S, T , and a family of t sets containing S but disjoint from T , and the other family of another t sets containing T but disjoint from S . This theorem echoes our statement in the abstract.

The result for $t = 2$ and $k = 2$ was proven in [1] and many details worked out for $t = 2$ and $k > 2$ by the first author and Peter Keevash. The extension for $t > 2$, $k = 2$ had been open since then [5]. The proof for $t > 2$, $k = 2$ is in Section 2. The proof for $t > 2$, $k > 2$ is in Section 3. Matrices $F_6(t), F_7(t)$ were given in [5] as 4-rowed forbidden configurations (with some columns of multiplicity t) for which Conjecture 1.2 predicts $\text{forb}(m, F_6(t))$ and $\text{forb}(m, F_7(t))$ are $O(m^2)$. Note that $t \cdot F_{0,2,2,0} \prec F_6(t)$ and $t \cdot F_{0,2,2,0} \prec F_7(t)$ and so Theorem 1.3 is a step towards these bounds which would establish Conjecture 1.2 for all 4-rowed F . Our proof use a new induction given in Section 2 that considers t -simple matrices as well as a strong stability result Lemma 3.5. We offer some additional applications in Section 4.

2 New Induction

We consider a new form of the standard induction for forbidden configurations [5]. Let F be a matrix with maximum column multiplicity t . Thus $F \prec t \cdot \text{supp}(F)$. Let $A \in \text{Avoid}(m, F, t-1)$. Assume $\|A\| = \text{forb}(m, \mathcal{F}, t-1)$. Given a row r we permute rows and columns of A to obtain

$$A \xrightarrow{\text{row } r} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ & G & & & H & & & \end{bmatrix}. \quad (2)$$

Now $\mu(\alpha, G) \leq t-1$ and $\mu(\alpha, H) \leq t-1$. For those α for which $\mu(\alpha, [GH]) \geq t$, let C be formed with $\mu(\alpha, C) = \min\{\mu(\alpha, G), \mu(\alpha, H)\}$. We rewrite our decomposition of A as follows:

$$A \xrightarrow{\text{row } r} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ B & C & & C & D & & & \end{bmatrix}. \quad (3)$$

Then we deduce that $[BCD]$ and C are both $(t-1)$ -simple. The former follows from $\mu(\alpha, [BCD]) = \mu(\alpha, G) + \mu(\alpha, H) - \min\{\mu(\alpha, G), \mu(\alpha, H)\} \leq t-1$. We have that $F \not\prec [BCD]$ for $F \in \mathcal{F}$. Also for any $F' \prec C$ then $[01] \times F' \prec A$ so we define

$$\mathcal{G} = \{F' : \text{for } F \in \mathcal{F}, F \prec [01] \times F' \text{ and } F \not\prec [01] \times F'' \text{ for all } F'' \prec F', F'' \neq F'\}. \quad (4)$$

Also since each column α of C has $\mu(\alpha, [GH]) \geq t$, we deduce that $\text{supp}(F) \not\prec C$ for each $F \in \mathcal{F}$. Our induction on m becomes:

$$\begin{aligned} \text{forb}(m, \mathcal{F}, t-1) &= \|A\| = \|[BCD]\| + \|C\| \\ &\leq \text{forb}(m-1, \mathcal{F}, t-1) + (t-1) \cdot \text{forb}(m-1, \mathcal{G} \cup \{\text{supp}(F) : F \in \mathcal{F}\}). \end{aligned} \quad (5)$$

Proof of Theorem 1.3 for $k = 2$: We will use induction on m to show $\text{forb}(m, t \cdot F_{0,2,2,0}, t)$ is $O(m^2)$. The maximum multiplicity of a column in $t \cdot F_{0,2,2,0}$ is t and $F_{0,2,2,0} = \text{supp}(t \cdot F_{0,2,2,0})$. Also $t \cdot F_{0,2,2,0} \prec [01] \times (t \cdot F_{0,2,1,0})$. Let $A \in \text{Avoid}(m, t \cdot F_{0,2,2,0}, t-1)$ with $\|A\| = \text{forb}(m, t \cdot F_{0,2,2,0}, t-1)$. Apply (5). We have

$$\begin{aligned} \text{forb}(m, t \cdot F_{0,2,2,0}, t-1) &= \|A\| = \|[BCD]\| + \|C\| \\ &\leq \text{forb}(m-1, t \cdot F_{0,2,2,0}, t-1) + (t-1) \cdot \text{forb}(m-1, \{F_{0,2,2,0}, t \cdot F_{0,2,1,0}\}). \end{aligned}$$

We apply Lemma 2.1 with induction on m to deduce that $\text{forb}(m, t \cdot F_{0,2,2,0}, t-1)$ is $O(m^2)$. Then by (1), $\text{forb}(m, t \cdot F_{0,2,2,0})$ is also $O(m^2)$. ■

Theorem 1.3 was proven for $t = k = 2$ in [1] using induction in the spirit (5) ($(t-1)$ -simple matrices are simple) and Lemma 2.1 for $t = 2$.

Lemma 2.1 *We have that $\text{forb}(m, \{F_{0,2,2,0}, t \cdot F_{0,2,1,0}\})$ is $O(m)$.*

Proof: Let $A \in \text{Avoid}(m, \{F_{0,2,2,0}, t \cdot F_{0,2,1,0}\})$. Avoiding $F_{0,2,2,0}$ creates structure: Let X_i denote the columns of A of column sum i . Let $J_{a \times b}$ denote the $a \times b$ matrix of 1's and let $0_{a \times b}$ denote the $a \times b$ matrix of 0's. Now $F_{0,2,2,0} \not\prec X_i$ and so for $\|X_i\| \geq 3$, we may deduce that there is a partition of the rows $[m]$ into $A_i \cup B_i \cup C_i$. Let $x_i = |X_i|$. After suitable row and column permutations, we have X_i as follows:

$$\text{type 1: } X_i = \begin{bmatrix} A_i \{ I_{x_i} \\ B_i \{ J_{(i-1) \times x_i} \\ C_i \{ 0_{(m-x_i-i+1) \times x_i} \end{bmatrix} \quad \text{or type 2: } X_i = \begin{bmatrix} A_i \{ I_{x_i}^c \\ B_i \{ J_{(i-x_i+1) \times x_i} \\ C_i \{ 0_{(m-i-1) \times x_i} \end{bmatrix}.$$

We will say i is of type j ($j = 1$ or $j = 2$) if the columns of sum i are of type j . These are the *sunflowers* (for type 1) and *inverse sunflowers* (type 2) of [7] where for type 1 the petals are A_i with center B_i .

Let $T(1) = \{i : i \text{ is of type 1 and } \|X_i\| \geq t+2\}$. We wish to show for that $B_i \subset B_j$ for $i, j \in T(1)$ and $i < j$. Assume $p \in B_i \setminus B_j$. Given that $|B_i| < |B_j|$, there are two rows $r, s \in B_j \setminus B_i$. Then we find a copy of $t \cdot F_{0,2,1,0}$ in rows p, r, s of $[X_i X_j]$ (we would not choose the possible column of X_i that has a 1 in row r and the column of X_i that has a 1 in row s), a contradiction showing no such p exists and hence $B_i \subset B_j$.

We form a matrix Y_1 from those X_i with $i \in T(1)$. We have $\|Y_1\| = \sum_{i \in T(1)} \|X_i\| = \sum_{i \in T(1)} |A_i|$. Assume $\sum_{i \in T(1)} |A_i| > (t+1)m$. Then there is some row p and $(t+2)$ -set $\{s(1), s(2), \dots, s(t+2)\}$ with $p \in A_i$ for all $i \in \{s(1), s(2), \dots, s(t+2)\}$. Assume $s(1) < s(2) < \dots < s(t+2)$. We have $B_{s(1)} \subset B_{s(2)} \subset \dots \subset B_{s(t+2)}$. We may choose $r, s \in B_{s(t+2)} \setminus B_{s(t)}$ so that $r, s \in A_{s(i)} \cup C_{s(i)}$ for $i = 1, 2, \dots, t$. We find a copy of $t \cdot F_{0,2,1,0}$ in rows p, r, s as follows. We take one column from each $X_{s(j)}$ for $j = 1, 2, \dots, t$ and t columns from the $X_{s(t+2)}$. We conclude that $\|Y_1\| \leq (t+1)m$. Similarly the matrix Y_2 formed from those X_i such that i is of type 2 and $\|X_i\| \geq t+2$ has $\|Y_2\| \leq (t+1)m$. Now Y_1 and Y_2 represent all columns of A with the exception of columns of sum i with $\|X_i\| \leq t+1$ and so we conclude $\|A\| \leq \|Y_1\| + \|Y_2\| + (t+1)(m-1) + 2$. Thus $\|A\|$ is $O(m)$. ■

3 More evidence for the Conjecture

This section first explores the Conjecture 1.2 for $t \cdot F$ when F is $k \times 2$. The section concludes with the proof of Theorem 1.3 for $k > 2$. The following verifies Conjecture 1.2 for all $k \times 2$ F . Note that any $k \times 2$ matrix F can be written as $F_{a,b,c,d}$ ($b \geq c$) under proper row and column permutations. Since $\text{forb}(m, F)$ is invariant under taking $(0, 1)$ -complement, we can further assume $a \geq d$. The case of $t = 1$ was solved in [3] by the following theorem.

Theorem 3.1 [3] *Suppose $a \geq d$ and $b \geq c$. Then $\text{forb}(m, F_{a,b,c,d})$ is $\Theta(m^{a+b-1})$ if either $b > c$ or $a, b \geq 1$. Also $\text{forb}(m, F_{a,0,0,d})$ is $\Theta(m^a)$ and $\text{forb}(m, F_{0,b,b,0})$ is $\Theta(m^b)$.* ■

Note that Conjecture 1.2 is verified if there is a product construction avoiding F yielding the same asymptotic growth as an upper bound on $\text{forb}(m, F)$. The k -fold product $I_{m/k} \times I_{m/k} \times \cdots \times I_{m/k} \in \text{Avoid}(m, t \cdot F_{0,k,k,0})$ has $\Theta(m^k)$ columns. Thus Theorem 1.3 verifies the conjecture for $t \cdot F_{0,k,k,0}$. The following results verify the conjecture for $t \cdot F$ for the remaining $k \times 2$ F .

Theorem 3.2 *For $b > c$ or $a, b \geq 1$ then $\text{forb}(m, t \cdot F_{a,b,c,d})$ is $\Theta(m^{a+b})$.*

Proof: The upper bound follows from $\text{forb}(m, F_{a,b,c,d})$ being $\Theta(m^{a+b-1})$ and then applying Lemma 4.1. The lower bound follows from $2 \cdot \mathbf{1}_{a+b} \prec t \cdot F_{a,b,c,d}$ so that the $(a+b)$ -fold product $I_{m/(a+b)} \times I_{m/(a+b)} \times \cdots \times I_{m/(a+b)} \in \text{Avoid}(m, F_{a,b,c,d})$ and hence $\text{forb}(m, t \cdot F_{a,b,c,d})$ is $\Omega(m^{a+b})$. ■

Theorem 3.3 *Let $a \geq d$ be given. Then $\text{forb}(m, t \cdot F_{a,0,0,d})$ is $\Theta(m^a)$.*

Proof: This follows using Lemma 3.4 repeatedly and also $\text{forb}(m, t \cdot F_{a,0,0,0})$ is $O(m^a)$ using Theorem 4.3. The a -fold product $I_{m/a} \times I_{m/a} \times \cdots \times I_{m/a} \in \text{Avoid}(m, t \cdot F_{a,0,0,d})$. ■

The following result can be found in the survey on forbidden configurations [5]

Lemma 3.4 *Assume $\text{forb}(m, F)$ is $O(m^\ell)$. Then $\text{forb}(m, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \times F)$ is $O(m^{\ell+1})$.*

Here is the summary of results on $\text{forb}(m, t \cdot F_{a,b,c,d})$ ($a \geq d$ and $b \geq c$), which verify Conjecture 1.2 for all $k \times 2$ F .

t	Configuration	result	reference	Lower bound construction
$t = 1$	$F_{a,b,c,d}$ ($b > c$ or $a, b \geq 1$)	$\Theta(m^{a+b-1})$	[3]	$\overbrace{I \times I \times \cdots I \times I}^{a+b-1}$
	$F_{a,0,0,d}$	$\Theta(m^a)$	[3]	$\overbrace{I \times I \times \cdots I \times I}^a$
	$F_{0,b,b,0}$	$\Theta(m^b)$	[3]	$\overbrace{I \times I \times \cdots I \times I}^{b-1} \times T$
$t \geq 2$	$t \cdot F_{a,b,c,d}$ ($b > c$ or $a, b \geq 1$)	$\Theta(m^{a+b})$	Lemma 4.1	$\overbrace{I \times I \times \cdots I \times I}^{a+b}$
	$t \cdot F_{a,0,0,d}$	$\Theta(m^a)$	Lemma 4.1	$\overbrace{I \times I \times \cdots I \times I}^a$
	$t \cdot F_{0,b,b,0}$	$\Theta(m^b)$	Theorem 1.3	$\overbrace{I \times I \times \cdots I \times I}^{b-1} \times T$

Table 1: All cases of $\text{forb}(m, t \cdot F_{a,b,c,d})$ with $a \geq d$ and $b \geq c$.

We note that the bound for $\text{forb}(m, t \cdot F_{a,0,0,d})$ can be readily established by a pigeonhole argument. We return to Theorem 1.3 and first obtain some useful lemmas.

Let $X_i \in \text{Avoid}(m, F_{0,k,k,0})$ with all column sums i . We define X_i to be of *type* (a, b) if $a, b \geq 0$ are integers with $a + b = k - 1$ and there is a partition $C_i \cup D_i = [m]$ with $|D_i| + a - b = i$ such that any column α of X_i has exactly a 1's in rows C_i and exactly b 0's in rows D_i . We are able to use this structure in view of the following ‘strong stability’ result:

Lemma 3.5 [3] *Let $Y_i \in \text{Avoid}(m, F_{0,k,k,0})$ with all column sums i . Assume $\|Y_i\| \geq (6(k-1))^{5k+2} m^{k-2}$. Then there is an m -rowed submatrix X_i of Y_i and a pair of integers $a, b \geq 0$ with $a + b = k - 1$ such that X_i is of type (a, b) and where $\|Y_i\| - \|X_i\| \leq m^{k-3}$.*

Lemma 3.6 *Let $X_i \in \text{Avoid}(m, F_{0,k,k,0})$ have all columns of sum i and assume X_i is of type (a, b) with $a, b \geq 1$ with $a + b = k - 1$. Let $C_i \cup D_i = [m]$ be the associated partition of the rows. We form a bipartite graph $G_i = (V_i, E_i)$ with $V_i = \binom{C_i}{a} \cup \binom{D_i}{b}$ where we have $(C, D) \in E_i$ if there is a column of X_i with a 1's in rows C and $D_i \setminus D$ and b 0's in rows D and $C_i \setminus C$. Assume $|E_i| \geq 2km^{k-2}$. Then there is subgraph $G'_i = (V'_i, E'_i)$ of G_i with $|E'_i| \geq \frac{1}{2}|E_i|$ such that for every pair $C \in \binom{C_i}{a}$ and $D \in \binom{D_i}{b}$ with $(C, D) \in E'$ we have*

$$d_{G'_i}(C) \geq (b + 1/2)m^{b-1}, \quad d_{G'_i}(D) \geq (a + 1/2)m^{a-1}. \quad (6)$$

Proof: Simply delete vertices $C \in \binom{C_i}{a}$ with $d_G(C) < (b + 1/2)m^{b-1}$ and vertices $D \in \binom{D_i}{b}$ with $d_G(D) < (a + 1/2)m^{a-1}$ and continue deleting vertices until conditions (6) are satisfied for any remaining vertices of G' . This will delete a maximum of $(b + 1/2)m^{b-1} \binom{|C_i|}{a} + (a + 1/2)m^{a-1} \binom{|D_i|}{b} < km^{k-2}$ edges which deletes less than half the edges of G . ■

Lemma 3.7 *Let k be given. Then $\text{forb}(m, \{F_{0,k,k,0}, t \cdot F_{0,k,k-1,0}\})$ is $O(m^{k-1})$.*

Proof: Let $A \in \text{Avoid}(m, \{F_{0,k,k,0}, t \cdot F_{0,k,k-1,0}\})$. Let Y_i denote the columns of A of column sum i . For all i for which $|Y_i| < (6(k-1))^{5k+2} m^{k-2}$, delete the columns of Y_i from A . This may delete $(6(k-1))^{5k+2} m^{k-1}$ columns. For i with $|Y_i| \geq (6(k-1))^{5k+2} m^{k-2}$, apply Lemma 3.5 and obtain X_i with $|X_i| \geq (6(k-1))^{5k+2} m^{k-2} - m^{k-3}$.

We consider a choice a, b with $a + b = k - 1$. Let $T(a, b) = \{i : X_i \text{ is of type } (a, b)\}$. We will show that $\sum_{i \in T(a,b)} |X_i| \leq (tk)m^{k-1}$.

Case 1. $a, b \geq 1$.

Create G_i as described in Lemma 3.6 to obtain G'_i for each $i \in T(a, b)$. Now if $\sum_{i \in T(a,b)} |E'_i| > (t+1)m^{a+b}$, then there will be some edge $(C, D) \in E'_i$ for at least $t+2$ choices $i \in T(a, b)$. Let those choices be $s(1), s(2), \dots, s(t+2)$ where $s(1) < s(2) < \dots < s(t+2)$. We wish to show that $X_{s(i)}$ has $t \cdot F_{0,k-1,0,0}$ on rows $C \cup D$.

$$\begin{array}{l} \text{rows } C \left\{ \begin{array}{l} 1 \quad \overbrace{11 \dots 1}^t \quad \overbrace{00 \dots 0}^t \\ 1 \quad 11 \dots 1 \quad 00 \dots 0 \\ 1 \quad 11 \dots 1 \quad 00 \dots 0 \end{array} \right. \\ \text{rows } D \left\{ \begin{array}{l} 0 \quad 11 \dots 1 \quad 00 \dots 0 \\ 0 \quad 11 \dots 1 \quad 00 \dots 0 \end{array} \right. \end{array}$$

For a given set $D \in \binom{D_{s(i)}}{b}$, we compute $|\{H \in \binom{D_{s(i)}}{b} : H \cap D \neq \emptyset\}| \leq \sum_{j=1}^b \binom{b}{j} \binom{D_{s(i)} \setminus D}{b-j} < bm^{b-1}$.

Now if $d_{G'}(C) \geq (b + 1/2)m^{b-1}$ and $(C, D) \in E'_{s(i)}$ then there are at least t edges $(C, H) \in E'_{s(i)}$ with $H \cap D = \emptyset$. We are using $(b + 1/2)m^{b-1} > bm^{b-1} + t + 2$ which is true for m large enough and so asymptotics are unaffected. Thus we have t columns of $X_{s(1)}$ with $\mathbf{1}_{k-1}$ on rows $C \cup D$ and, because these columns have a 1's on rows $C \subseteq C_{s(1)}$, these columns are 0's on the remaining rows of $C_{s(1)} \setminus C$.

Similarly, because $d_{G'_i}(D) \geq (a + 1/2)m^{a-1}$ there will be $t + 2$ edges $(K, D) \in E_{s(i)}$ with $K \cap C = \emptyset$ and so there are t columns of $X_{s(t+2)}$ with $\mathbf{0}_{k-1}$ on rows $C \cup D$ and, because these columns have 0's on rows D , these columns are 1's on rows of $D_{s(t+2)} \setminus D$.

We choose k rows in $Z = D_{s(t+2)} \setminus D_{s(1)}$ so that $Z \subseteq C_{s(1)}$. We deduce that in the chosen t columns of $X_{s(1)}$ we have $\mathbf{0}_k$ in rows Z since $Z \subseteq C_{s(1)} \setminus C$ and the columns have $\mathbf{1}_{k-1}$ in rows $C \cup D$. In the chosen t columns of $X_{s(t+2)}$ we have $\mathbf{1}_k$ in rows Z since $Z \subset D_{s(t+2)} \setminus D$ and the columns have $\mathbf{0}_{k-1}$ in rows $C \cup D$. This yields $t \cdot F_{0,k,k-1,0}$, a contradiction. Thus $\sum_{i \in T_{\text{type}(a,b)}} |E'_i| \leq (t + 1)m^{k-1}$. This concludes Case 1.

Case 2. $a = k - 1, b = 0$ or $a = 0, b = k - 1$.

We proceed similarly. We need only consider $a = k - 1, b = 0$ since the case $a = 0, b = k - 1$ is just the $(0,1)$ -complement. For $i \in T(k - 1, 0)$, X_i has partition $C_i \cup D_i = [m]$ and columns of X_i have 1's on exactly $k - 1$ rows of C_i and all 1's on rows D_i . Assume $\sum_{i \in T(k-1,0)} |X_i| \geq (tk)m^{k-1}$. Then there are tk choices $s(1), s(2), \dots, s(tk) \in T(k - 1, 0)$ where $s(1) < s(2) < \dots < s(tk)$ such that, for some $C \in \binom{C_{s(i)}}{k-1}$, each $X_{s(i)}$ has a column with 1's in rows $C \cup D_{s(i)}$ and 0's in rows $C_{s(i)} \setminus C$. We wish to find $t \cdot F_{0,k-1,0,0}$ in A in rows C as follows using one column from each of $X_{s(i)}$ for $i = 1, 2, \dots, t$ and t columns from $X_{s(tk)}$.

$$\text{rows } C \left\{ \begin{array}{ccccc} 1 & 1 & & 1 & 1 & \overbrace{00 \dots 0}^t \\ 1 & 1 & \dots & 1 & 1 & 00 \dots 0 \\ 1 & 1 & & 1 & 1 & 00 \dots 0 \\ X_{s(1)} & X_{s(2)} & & X_{s(t)} & X_{s(tk)} & X_{s(tk)} \end{array} \right.$$

Given our choice $C \in \binom{C_{s(tk)}}{k-1}$, we compute that $|\{K \in \binom{C_{s(tk)}}{k-1} : K \cap C \neq \emptyset\}| < km^{k-2}$. Thus with $|X_{s(tk)}| \geq km^{k-2}$, there will be t choices K_1, K_2, \dots, K_t disjoint from C and hence one column of $X_{s(i)}$ for each $i = 1, 2, \dots, t$ with $\mathbf{1}_{k-1}$ on rows of $K_i \subseteq C_{s(i)} \setminus C$ and 0's on $C_{s(i)} \setminus K_i$ and hence $\mathbf{0}_{k-1}$ on rows C .

We will show below that we can choose $D \subset D_{s(tk)} \setminus \bigcup_{i=1}^t D_{s(i)}$ with $|D| = k$. Then we can find $t \cdot F_{0,k,k-1,0}$ as follows. We have one column in $X_{s(i)}$ for each $i = 1, 2, \dots, t$ which is $\mathbf{1}_{k-1}$ on rows C and $\mathbf{0}_k$ on rows D (since $D \subset C_{s(i)} \setminus C$ for each $i = 1, 2, \dots, t$). The t columns of $X_{s(tk)}$ we have selected have $\mathbf{0}_{k-1}$ on rows C and 1's on $D_{s(tk)}$ where $D \subseteq D_{s(tk)}$ and hence $\mathbf{1}_k$ on rows D . These $2t$ columns yield $t \cdot F_{0,k,k-1,0}$ in $[X_{s(1)} \mid X_{s(2)} \mid \dots \mid X_{s(t)} \mid X_{s(tk)}]$.

To show that D can be chosen we first show that $D_{s(i)} \setminus D_{s(j)} \leq k - 2$ for $s(i) < s(j)$. Assume the contrary, $D_{s(i)} \setminus D_{s(j)} \geq k - 1$ for $s(i) < s(j)$. We choose $C' \subseteq D_{s(i)} \setminus D_{s(j)}$ with $|C'| = k - 1$. Given $s(j) > s(i)$, then $D_{s(j)} \setminus D_{s(i)} \geq k$ and so we may choose $D' \subseteq D_{s(j)} \setminus D_{s(i)}$ with $|D'| = k$. Now $C' \subset C_{s(j)}$ and $D' \subset C_{s(i)}$. The number of possible columns of $X_{s(j)}$ with at least one 1 on the rows C' is at most m^{k-2} and with $|X_{s(j)}| \geq m^{k-1} + t$, we find t columns of $X_{s(j)}$ with 0's on rows C' and necessarily with 1's on rows D' . The number of possible columns of $X_{s(i)}$ with at least one 1 on the rows of D' is $|D'|m^{k-2} < m^{k-1}$. Given $|X_{s(i)}| \geq m^{k-1} + t$, we find t columns of $X_{s(i)}$ with 0's on rows D' and necessarily with 1's on rows C' . This yields $t \cdot F_{0,k,k-1,0}$ in $[X_{s(i)} | X_{s(j)}]$, a contradiction. Thus $D_{s(i)} \setminus D_{s(j)} \leq k - 2$ for $s(i) < s(j)$. We may now conclude that $|D_{s(kt)} \setminus \bigcup_{i=1}^t D_{s(i)}| \geq k$ and so a choice for D exists. We conclude $\sum_{i \in T(k-1,0)} |X_i| \leq (tk)m^{k-1}$. This concludes Case 2.

There are $k + 1$ choices for type (a, b) and so

$$\sum_{i=0}^m |X_i| \leq \sum_{j=0}^k \left(\sum_{i \in T(j, k-1-j)} |X_i| \right) \leq (k+1)(2tk)m^{k-1}$$

and so $\|A\| \leq (2tk(k+1))m^{k-1} + (6(k-1))^{5k+2}m^{k-2}$ which is $O(m^{k-1})$. ■

Proof of Theorem 1.3 for $k \geq 3$: We use (5) so that $\text{forb}(m, t \cdot F_{0,k,k,0}, t-1) \leq \text{forb}(m-1, t \cdot F_{0,k,k,0}, t-1) + (t-1)\text{forb}(m, \{F_{0,k,k,0}, t \cdot F_{0,k,k-1,0}\})$. Induction on m and Lemma 3.7 yields the bound. ■

4 Some applications of the Induction

Lemma 4.1 *Let H be a given simple matrix satisfying $\text{forb}(m, H)$ is $O(m^\ell)$. Then $\text{forb}(m, t \cdot H)$ is $O(m^{\ell+1})$.*

Proof: We use the induction (5) where $F = t \cdot H$ and $H = \text{supp}(F)$. Induction on m yields the desired bound. ■

Let K_k denote the $k \times 2^k$ of all possible (0,1)-columns on k rows. The following is the fundamental result about forbidden configurations.

Theorem 4.2 *[Sauer [10], Perles and Shelah [11], Vapnik and Chervonenkis [12]] We have that*

$$\text{forb}(m, K_k) = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}.$$

Thus $\text{forb}(m, K_k)$ is $\Theta(m^{k-1})$.

We can apply this result as follows.

Theorem 4.3 [8] *Let F be a given $k \times \ell$ $(0,1)$ -matrix. Then $\text{forb}(m, F)$ is $O(m^k)$.*

Proof: Let t be the maximum multiplicity of a column in F (of course $t \leq \ell$). Then $F \prec t \cdot K_k$ and so $\text{supp}(F) \prec K_k$. Now Lemma 4.1 combined with Theorem 4.2 yields the result. ■

Interestingly this yields the exact result for $\text{forb}(m, 2 \cdot K_k)$ [9]. A more precise result of Anstee and Füredi [2] for $\text{forb}(m, t \cdot K_k)$ has the leading term being bounded by $\frac{t+k-1}{k+1} \binom{m}{k}$ for $t \geq 2$. The following surprising result was obtained by Balogh and Bollobás.

Theorem 4.4 [6] *Let k be given. There is a constant c_k with $\text{forb}(m, \{I_k, I_k^c, T_k\}) = c_k$.*

This yields the following.

Theorem 4.5 *Let $t, k \geq 2$ be given. Then $\text{forb}(m, \{t \cdot I_k, t \cdot I_k^c, t \cdot T_k\})$ is $\Theta(m)$.*

Proof: Apply Lemma 4.1. The matrix $I_m \in \text{Avoid}(m, \{t \cdot I_k, t \cdot I_k^c, t \cdot T_k\})$ shows that $\text{forb}(m, \{t \cdot I_k, t \cdot I_k^c, t \cdot T_k\})$ is $\Theta(m)$. ■

Lemma 4.1 is interesting for those H for which $\text{forb}(m, H)$ is $O(m^\ell)$ and the number of rows in H is bigger than ℓ (see [5] for examples). It is not expected that this will resolve any *boundary cases*, namely those F for which $\text{forb}(m, [F | \alpha])$ is bigger than $\text{forb}(m, F)$ by a linear factor (or more) for all choices α which are either not present in F or occur at most once in F . The previously mentioned $F_6(t)$ and $F_7(t)$ have quite complicated structure and the induction (5) does not appear to work directly.

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